A New Notion of g-Angle in Normed

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A New Notion of *g-Angle* in Normed

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A New Notion of g-Angle in Normed

Muh Nur [1*], **Mawardi Bahri** [1], **and Anna Islamiyati**[2]
[1] Department of Mathematics , Hasanuddin University, Indonesia
[2] Department of Statistics, Hasanuddin University, Indonesia

*Email: muhammadnur@unhas.ac.id

Abstract. We discuss the g-orthogonality ($\perp^g$) and its basic properties in a normed space. We also define a new g-angle between two vectors and prove its basic properties by using the g-orthogonality ($\perp^g$). Moreover, we prove the sufficient and necessary conditions for strictly convexity in a normed space by using a new g-angle.

1. Introduction

The notions of orthogonality and angle in inner product space $(X, \langle\cdot,\cdot\rangle)$ are interesting and old mathematical theme. The angle $B(u,v)$ in inner product space defined by $B(u,v) \coloneqq \arccos \frac{\langle u,v\rangle}{\|u\|\times\|v\|}$. Here, $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$. The angle $B(u,v)$ satisfies the following properties.

1) u and v are of the same direction if and only if $B(u,v) = 0$;
   u and v are of opposite direction if and only if $B(u,v) = \pi$.
2) $B(u,v) = B(v,u)$.
3) $B(\alpha u, \beta v) = \begin{cases} B(u,v), & \text{if } \alpha\beta > 0 \\ \pi - B(u,v), & \text{if } \alpha\beta < 0. \end{cases}$
4) If $u_n \to u$ and $v_n \to v$ (in norm) then $B(u_n, v_n) \to B(u,v)$. (see [1]).

Then u is said to be orthogonal to v $(u \perp v)$ if only if $\langle u,v\rangle = 0$, i.e., $B(u,v) = \pi/2$. Several notions of orthogonality and angle between two vector also have been developed in normed space with the norm not necessarily coming from an inner product during last decades.— see, for example [2, 3, 4, 5, 6, 7, 8,9].

Milićić [10,11,12] introduced g-orthogonality in normed spaces $(X, \|\cdot\|)$ via Gateaux derivatives. The mapping $g: X^2 \to \mathbb{R}$ defined by

$$g(u,v) \coloneqq \frac{1}{2}\|u\|[\phi_+(u,v) + \phi_-(u,v)]$$

where

$$\phi_\pm(u,v) \coloneqq \lim_{i\to 0^\pm} \frac{\|u+iv\| - \|u\|}{i}. \quad (1)$$

Then, one may check that g satisfies the following fact :

1) $g(u,u) = \|u\|^2$;
2) $g(\alpha u, \beta v) = \alpha\beta g(u,v)$ for all $\alpha, \beta \in \mathbb{R}$;
3) $g(u, u+v) = \|u\|^2 + g(u,v)$;
4) $|g(u,v)| \leq \|u\|\|v\|$.

The maping $g(u,v)$ is a *semi-inner product* on X if, in addition, $g(u,v)$ is linear in v. For instance, for every $u, v \in l^p$ $(1 \leq p < \infty)$, $g(u,v)$ is a semi-inner product [13,14] with

$$g(u,v) = \|u\|_p^{2-p} \sum_j |u_j|^{p-1} \operatorname{sgn}(u_j) v_j\,, \quad u \coloneqq (u_j), v \coloneqq (v_j) \in l^p\,. \tag{2}$$

Milićić [11] defined the g-orthogonal by $u \perp_g v$ provided $g(u,v) = 0$. Related to g-orthogonality, the g-angle $B_g(u,v)$ defined by $B_g(u,v) = \arccos \frac{g(u,v)}{\|u\| \times \|v\|}$. Note that in $(X, \langle \cdot,\cdot \rangle)$, $B_g(u,v)$ is identical with $B(u,v)$.

In this paper, we will discuss the other g-orthogonality $(\perp^g)$ in a normed space that was also introduced Milićić. Next, we will introduce a new g-angle $B^g(u,v)$ and discuss its connection to $B_g(u,v)$. In the last section, we prove the sufficient and necessary conditions for strictly convexity in a normed space by using a new g-angle.

2. Main result

2.1. The g-orthogonality two vectors $(\perp^g)$

In this subsection, we can discuss the g-orthogonality $(\perp^g)$ in $(X, \|\cdot\|)$ and its basic properties. Firstly, we write the mapping $(\cdot,\cdot)^g$ on X by

$$(u,v)^g \coloneqq g(u,v) \times g(v,u) \tag{3}$$

where g is the semi-inner product on X. By using properties of g presented in the introduction, we obtain the following result.

Fact 1. The function (3) satisfies the following conditions:
(1) $(u,u)^g = \|u\|^4$ for any $u \in X$;
(2) $(u,v)^g = (v,u)^g$ for any $u, v \in X$ and $\alpha, \beta \in \mathbb{R}$; ;
(3) $(\alpha u, \beta v)^g = (\alpha\beta)^2 \times (u,v)^g$ for any $u, v \in X$ and $\alpha, \beta \in \mathbb{R}$;
(4) $|(u,v)^g| \leq \|u\|^2 \times \|v\|^2$ for any $u, v \in X$.

Remark 2. The property of the triangle inequality is not fulfilled. For instance, consider the normed space $(l^1, \|\cdot\|_1)$. Take $u = (1, -2, 0, \dots), v = (0, 2, 0, \dots)$, and $w = (3, 1, 0, \dots)$ in l^1. We observe that $g(u,w) = 8$, $g(v,w) = 2$, $g(w,u) = 0$, $g(w,v) = 8$, $g(w, u+v) = 8$ and $g(u+v, w) = 6$. Therefore, $(u+v, w)^g > (u,w)^g + (v,w)^g$.

Using the mapping $(\cdot,\cdot)^g$ in (3), we write the other g-orthogonality that was introduced by Milićić [11] as follows.

Definition 3. [11] Vector u is said to be *g-orthogonal* to vector v in a normed space, and we write $u \perp^g v$ if the mapping $(u,v)^g = 0$.

In inner product space, $u \perp^g v$ is identical with $\langle u, v \rangle = 0$. By using Fact. 1 and semi-inner product, we obtain the properties the g-orthogonality $(\perp^g)$ as follows.

Theorem 4. *The g-orthogonality $(\perp^g)$ satisfies the following conditions:*
(1) $u \perp^g u$ implies $u = 0$ (nondegeneracy property).
(2) $u \perp^g v$ implies $v \perp^g u$ (symmetry property).
(3) $u \perp^g v$ implies $au \perp^g bv$ for any $a, b \in \mathbb{R}$ (homogeneity property).
(4) For all $u, v \in X$ there is a real number γ such that $u \perp^g (\gamma u + v)$ (resolvability property).

Proof. The properties $(1) - (3)$ are obviously true by Fact 1. Next, for (4), choose $\gamma = -\frac{g(u,v)}{\|u\|^2}$. Then, we obtain

$$\begin{aligned}(u, \gamma u + v)^g &= g(u, \gamma u + v) \times g(\gamma u + v, u)\\ &= \big(\gamma g(u,v) + g(u,v)\big) \times g(\gamma u + v, u) = 0,\end{aligned}$$

as desired. ■

Next, we have the connection between two g-orthogonality ($\perp^g$ and $\perp_g$) as follows.

Proposition 5. *For every $u, v \in X$,* if $u \perp_g v$ then $u \perp^g v$.

Remark 6. Converse of the above proposition is not fulfilled. For instance in $(l^1, \|\cdot\|_1)$, take $u = (2,2,0, \dots), v = (-2,3,0, \dots)$. Using properties of g, we can observe that $g(u,v) = 4$ but $(u,v)^g = 0$.

Next, we obtain result as follows.

Proposition 7. *Let $(X, \|\cdot\|)$ be a normed space. Then*
(1) $u \perp_g v$ and $u \perp_g v'$ implies $u \perp^g v + v'$ for any $u, v, v' \in X$.
(2) $v_n \to v$ (in norm) and $u \perp_g v_n$ for any $n \in \mathbb{N}$ implies $u \perp^g v$.

Proof. (1) Because $u \perp_g v$ and $u \perp_g v'$ then $g(u,v) = 0$ and $g(u,v') = 0$. By properties of g, we obtain

$$\begin{aligned}(u, v + v')^g &= g(u, v + v') \times g(v + v', u)\\ &= (g(u,v) + g(u,v')) \times g(v + v', u) = 0.\end{aligned}$$

(2) By properties of g and $u \perp_g v_n$, we have

$$\begin{aligned}(u,v)^g &= g(u,v) \times g(v,u)\\ &= \big(g(u,v) - g(u,v_n)\big) \times g(v,u)\\ &= g(u, v - v_n) \times g(v,u)\\ &\le \|u\| \times \|v_n - v\| \times g(v,u).\end{aligned}$$

Because $\|v_n - v\| \to 0$, we obtain $u \perp^g v$. This proves the proposition.

2.2. The new g-angle $(B^g(\cdot,\cdot))$

In this subsection, we can introduce the new g-angle $(B^g(\cdot,\cdot))$ which preserves the above g-orthogonality. Define *the new* g-*angle* between two nonzero vectors $u, v \in (X, \|\cdot\|)$, denoted by $B^g(u,v)$ with

$$B^g(u,v) \coloneqq \arccos \frac{(u,v)^g}{\|u\|^2 \times \|v\|^2}.$$

Remark 8. For instance, consider $(l^1, \|\cdot\|_1)$, with $g(u,v) = \|u\|_1 \sum_j \operatorname{sgn}(u_j) v_j$. Take $u = (2,1,0,0, \dots)$ and $v = (3,-2,0,0, \dots)$. We observe that $\|u\|_1 = 3$, $\|v\|_1 = 5$, $g(v,u) = 5$ and $g(u,v) = 3$. So, we conclude that $B^g(u,v) = \arccos \frac{1}{15}$.

Note that $B^g(u,v) = \frac{1}{2}\pi$ if and only if $(u,v)^g = 0$. In $(X, \langle\cdot,\cdot\rangle)$, $B^g(\cdot,\cdot)$ is equivalent the usual angle $B(\cdot,\cdot)$.

Next, we have result as follows.

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Theorem 9. *The new g-angle $B^g(\cdot,\cdot)$ satisfies the following conditions:*

(1) For all $u, v \in X$, u and v are linearly dependent implies $B^g(u, v) = 0$.

(2) For all $u, v \in X$, $B^g(u, v) = B^g(v, u)$.

(3) *For all $u, v \in X$ and $\alpha, \beta \in \mathbb{R} - \{0\}$, $B^g(\alpha u, \beta v) = B^g(u, v)$.*

Proof.

(1) Suppose $u = cv$ for every $y \in X$ and $c \in \mathbb{R} - \{0\}$. Observe that

$$B^g(u, v) = \arccos\frac{(cv, v)^g}{\|cv\|^2 \times \|v\|^2} = \arccos\frac{c^2(v, v)^g}{c^2\|v\|^4} = 0.$$

(2) By using the properties of norm and Fact 1, we observe that

$$B^g(u, v) = \arccos\frac{(u, v)^g}{\|u\|^2 \times \|v\|^2}$$
$$= \arccos\frac{(v, u)^g}{\|v\|^2 \times \|u\|^2} = B^g(v, u).$$

(3) Suppose $\alpha, \beta \in \mathbb{R} - \{0\}$. Using the properties of norm and Fact 1, we have

$$B^g(\alpha u, \beta v) = \arccos\frac{(\alpha\beta)^2 \times (u, v)^g}{(\alpha\beta)^2 \times (\|u\|^2 \times \|v\|^2)}$$
$$= B^g(u, v).$$

as desired. ∎

Remark 10. The new angle $B^g(u, v)$ does not satisfy paralelisme and continuity property. For example, let $(l^1, \|\cdot\|_1)$ be a normed space with $g(u, v)$ in (1). Take $u = (-1, 1, 0, \dots)$ and $v = (1, -1, 0, \dots)$. We can see that x and y are linearly independent, but $B^g(u, v) = 0$. Next, with choose $u_n = (1 + \frac{1}{n}, 1, 0, \dots)$, $v_n = (\frac{1}{n}, 1, 0, \dots)$, u $(1, 1, 0, \dots)$ and $v = (0, 1, 0, \dots)$. We observe that $u_n \to u$ and $v_n \to v$ but $(u_n, v_n)^g \nrightarrow (u, v)^g$. Hence, $B^g(u_n, v_n) \nrightarrow B^g(u, v)$. By (3), we observe that $B^g(\cdot,\cdot)$ between two vectors is also $B^g(\cdot,\cdot)$ between two lines.

Next, we have the connection between the g-angle presented in the introduction $B_g(u, v)$ and the new g-angle $B^g(u, v)$ as follows.

Proposition 11. *Let $u, v \in X - \{0\}$.*

(1), $B^g(u, v) = 0$ *implies* $B_g(u, v) = 0$.

(2) $B_g(u, v) = \frac{\pi}{2}$ *implies* $B^g(u, v) = \frac{\pi}{2}$.

Proof.

(1) Suppose that $B^g(u, v) = 0$. By definition B^g, we obtain $g(u, v) \times g(v, u) = \|u\|^2 \times \|v\|^2$. Because $g(u, v) \le \|u\| \times \|v\|$ and $g(v, u) \le \|u\| \times \|v\|$, then $g(u, v) = \|u\| \times \|v\|$. Hence, $B_g(u, v) = 0$.

(2) Because $B_g(u,v) = \frac{\pi}{2}$, we obtain $u \perp_g v$. By using Proposition 5, we obtain $u \perp^g v$. Hence, we conclude that $B^g(u,v) = \frac{\pi}{2}$. ■

2.3. The connection between the new g-angle and strictly convex

In this subsection, we will prove the sufficient and necessary conditions for strictly convexity using the new g-angle. X is strictly convex if the following condition holds: if $\|u\| + \|v\| = \|u+v\|$ where $u, v \in X - \{0\}$, then $v = \tau u$ for some $\tau \in \mathbb{R}^+$. To prove the sufficient and necessary conditions for strictly convexity, we use the following theorem.

Theorem 12. *[13] A semi-inner product* $(X, [\cdot,\cdot])$ *is strictly convex if and only if this condition holds: for all* $u, v \in X - \{0\}$, *if* $[u,v] = \|u\| \times \|v\|$ *then* $v = \tau u$ *for some* $\tau \in \mathbb{R}^+$.

Finally, we have the result as follows.

Theorem 13. *X is strictly convex if and only if this condition holds:,* $\cos B^g(u,v) = 1$ *for all* $u, v \in X - \{0\}$ *implies* $v = \tau u$ *for some* $\tau \in \mathbb{R}^+$.

Proof. Assume X is strictly convex. If $\cos B^g(u,v) = 1$ then $g(u,v) \times g(v,u) = \|u\|^2 \times \|v\|^2$. Because $g(u,v) \leq \|u\| \times \|v\|$ and $g(v,u) \leq \|u\| \times \|v\|$, then $g(u,v) = \|u\| \times \|v\|$. By using Theorem 12, we obtain there is $\tau \in \mathbb{R}^+$such that $v = \tau u$.

Conversely, suppose that for every $u, v \in X - \{0\}$, $\cos B^g(u,v) = 1$ *implies* $v = \tau u$ *for some* $\tau \in \mathbb{R}^+$ hold. Consider $g(u,v) = \|u\| \times \|v\|$ for every $u, v \in X - \{0\}$. By using [15], we obtain $g(v,u) = \|u\| \times \|v\|$. As a consequence, for every $u, v \in X - \{0\}$, $\cos B^g(u,v) = 1$. By hypothesis, there is $\tau \in \mathbb{R}^+$such that $v = \tau u$. Hence, for every $u, v \in X - \{0\}$, $g(u,v) = \|u\| \times \|v\|$ implies there is $\tau \in \mathbb{R}^+$such that $v = \tau u$. By using Theorem 12, we obtain $(X, \|\cdot\|)$ is strictly convex. ■

3. Concluison

Based result has been given on above section, we have discussed the other g-orthogonality ($\perp^g$) and the new g-angle ($B^g(\cdot,\cdot)$). Next, we have proven basic properties. We also have shown the connection between the g-angle two vectors $\left(B_g(u,v)\right)$ and the new g-angle two vectors $\left(B^g(u,v)\right)$. Moreover, we have proven the sufficient and necessary conditions for strictly convexity in a normed space by using $B^g(u,v)$.

4. Acknowledgments

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5. References

[1] Diminnie C R, Andalafte E Z and Freese R W 1986 *Math. Nachr.* **129** 197–204.
[2] Alonso J, Martini H and Wu S 2012 *Aequat. Math.* **83** 153–189 .
[3] Gunawan H, Lindiarni J and Neswan O 2008 *J. Math. Fund. Sci.* **40(1)** 24–32.
[4] Milićić P M 1987 *Mat. Vesnik.* **39** 325–334.
[5] Balestro V, Horváth A G, Martini H and Teixeira R 2017 *Aequat. Math.* **91(2)** 201–236.
[6] Zhi-Zhi C, Wei L, Lu-Lin L, and You-qing J 2011 *Comm. Math. Res.* **27(4)** 378–384.
[7] Nur M and Gunawan H 2019 *Aequat. Math.* **93** 547–555.
[8] Nur M and Gunawan H 2021 *Aequat. Math.* **95** 309–318 (2021).

[9] Nur M, Gunawan H and Neswan O 2018 *Beitr. Algebra Geom.* **59(1)** 133–143.
[10] Milićić P M 1993 *Mat. Vesnik*, **45** 43–48.
[11] Milićić P M 1994 *Mathematica Montisnigri*, **45** 69–77.
[12] Milićić P M 2007 *J. Inequal. Pure and Appl. Math.* **8(3)** 1–9.
[13] Giles J R 1967 *Trans. Amer. Math Soc.* **129(3)** 436–446 (1967).
[14] Horváth A G 2010 *J. Geom. Phys.* **60** 1190–1208.
[15] Milićić P M 1998 *Mathematica Bilten*, **22**(XLVIII) 71–75.

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